Engineering Physics Semester I
Unit - II: Wave packet and wave equations

Wavepacket and its formation.

A single monochromatic wave cannot represent a particle.

Q1. A single monochromatic wave cannot represent a particle. Why? (2) S-09

Q2.Why does a single monochromatic wave not represent a localized particle? Explain synthesis of a wave packet. (4) S-04

Q3. Can a wave equation given by an equation Y=A sin (wt-kx) represent a particle? Explain the concept of wave packet. (3) S-00

Q4.What is wave packet? (2) S-17

Q5. What do you understand by wave packet? (2) W-15

Ans.

A single monochromatic wave can be represented by equation Y=A sin (ωt - kx) where 'k' is the propagation constant, ω is the angular frequency and 'A' is its amplitude. Such a wave extends infinitely in space. It has no beginning and no end. Such a single monochromatic wave cannot represent a particle since the particle is localized in space.

Wavepacket

Superposition of infinitely large number of waves of slightly different frequencies or wavelengths is known as wave packet. Such a wave packet can represent a matter wave associated with a particle.

Formation(Synthesis) of a wavepacket/ Concept of wavepacket

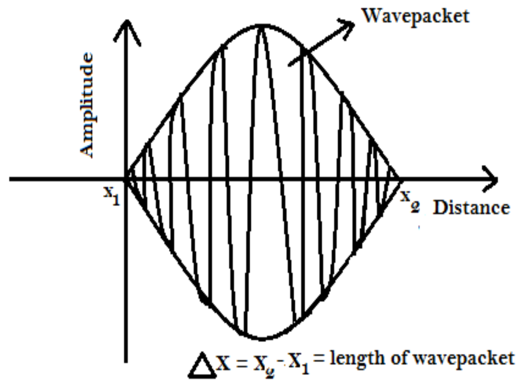


Figure 1: Wavepacket

As shown in figure1, wavepacket is a combination of several waves and has regular separation between successive maxima. **Hence it has all characteristics of a wave**. Since it is localized in space, it has all characteristics of particle. **Wavepacket has all the characteristics of a wave as well as particle.Therefore, a wavepacket can represent a particle and matter wave**. As shown in figure 1,the particle can be located anywhere between x_1 and x_2 inside the wavepacket but maximum probability of finding the particle is at the centre of this wave packet.

Q6.What do you understand by wave packet? Obtain relation between group velocity and phase velocity? (4) W-15

Ans**. Wavepacket**

Superposition of infinitely large number of waves of slightly different frequencies and wavelengths is known as wave packet. Such a wave packet can represent a matter wave associated with a particle.

Heisenberg's uncertainty principle.

Thought experiment on single slit electron diffraction.

Q7.State the Heisenberg's Uncertainty principle? (2)W-14, S-16

Q8. What is Heisenberg's Uncertainty principle? Explain how it is the outcome of the wave description of a particle. (4)W-04
Q9. Explain Heisenberg Uncertainty Principle with the aid of Thought experiment. (4) S-13, S-12
Q19. What is Uncertainty Principle? Is this principle the outcome of the wave description of a particle? Describe diffraction of electrons by single slit experiment to prove its validity.(5) W-13
Q20. What is Heisenberg's Uncertainty principle? Describe a thought experiment to arrive at this principle. (4) S-14, S-18,W-09,S-07,W-05
Q21.Explain a Thought experiment to arrive at Heisenberg Uncertainty Principle.(4)S-15
Q22. Arrive at Heisenberg Uncertainty Principle with the help of Thought experiment. (3) W-15, W-11,S-02
Q23.Discuss thought experiment of electron diffraction to arrive at Heisenberg Uncertainty Principle with suitable diagram. (4) S-17
Ans.
Heisenberg's Uncertainty principle: It is impossible to measure simultaneously both position and momentum of a particle with unlimited accuracy. Mathematically, if 'Δx' is the uncertainty in measurement of position of particle and 'Δp_x' is the uncertainty in measurement of its momentum then

$$\Delta x.\ \Delta p_x \geq \hbar.$$

Heisenberg's Uncertainty principle as the outcome of the wave description of a particle:
According to classical mechanics, one can measure both position and momentum of a body accurately at the same time. But in wave mechanics, a microparticle is described in terms of wavepacket. Inside this wavepacket, particle can be found anywhere.

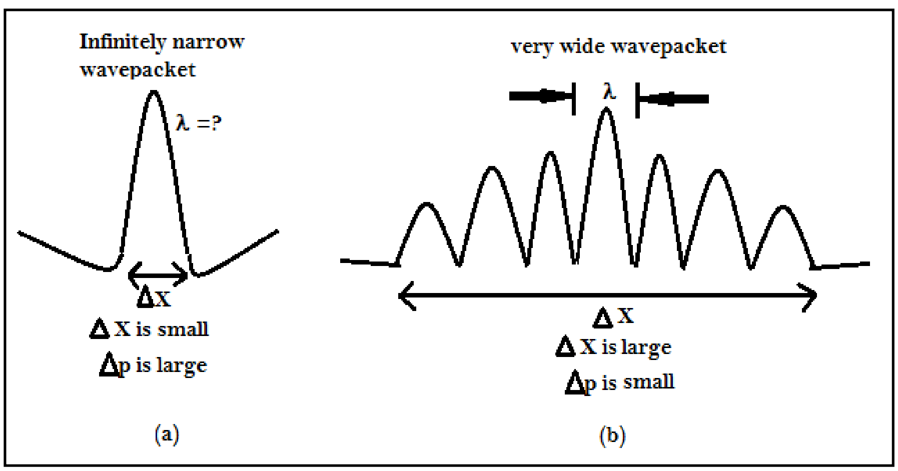


Figure2: Infinitely narrow and wide wavepacket.

As shown in figure 2(a) if the wavepacket is very narrow, the position of associated particle can be easily known but the measurement of its wavelength is impossible. Since $\lambda = \frac{h}{p}$ (by de Broglie hypothesis), it will also be difficult to measure momentum of the particle.Therefore, there is large uncertainty in measurement of momentum(Δp is large). On the other hand, if the wavepacket is wide as shown in figure 2(b), the wavelength can be measured accurately along with momentum but since the particle can be anywhere inside this large wavepacket, there is a large uncertainty in knowing exact position of the particle(Δx is large).
Realizing the consequences of this concept, Heisenberg developed his uncertainty principle-***"It is impossible to measure simultaneously both position and momentum of a particle with unlimited accuracy". Mathematically, product of uncertainty in measurement of position (Δx) and momentum (Δp_x) of a microparticle is always greater than or equal to Planck's constant h or ħ.***

$$\Delta x.\ \Delta p_x \geq \hbar.$$

Thought experiment :Diffraction of electrons by single slit: It is in Activity
Heisenberg uncertainty principle can be explained on the basis of a thought experiment- Diffraction of electrons by single slit.

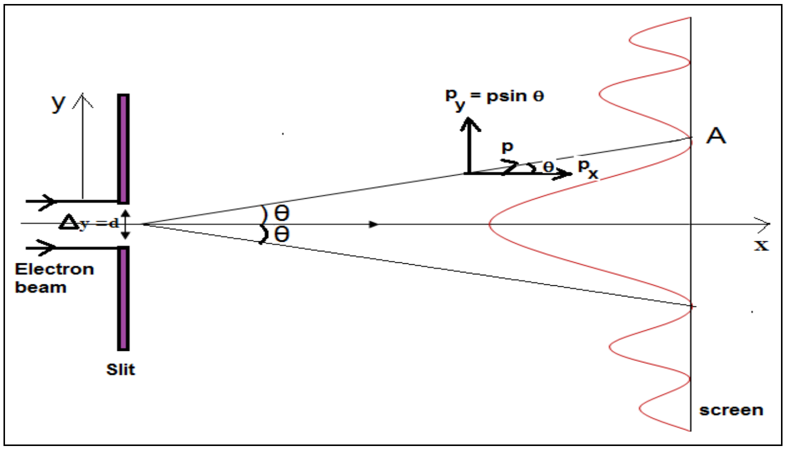


Figure3: Thought Experiment

Consider a monochromatic beam of electrons of wavelength 'λ' be moving along x-direction. Electrons are allowed to pass through the slit of width 'd' and produce diffraction pattern on a screen as shown in figure3.The condition for obtaining diffraction is nλ =d sinΘ.

For first minima, n=1, hence λ =d sinΘ ------- (1)

Before entering the slit, the electrons could only be located along x-direction. After passing through the slit, electrons can be located in y-direction and the maximum uncertainty in location of electrons in y-direction is Δy=d.

∴ From eqn. (1) we get $\sin\Theta = \frac{\lambda}{d} = \frac{\lambda}{\Delta y}$ --------- (2)

Before entering the slit there is no component of momentum along y direction. After diffraction the electrons acquire a component p_y in y direction.

The uncertainty in the y component of momentum is $\Delta p_y = p \sin\Theta$

$$\text{or } \sin\Theta = \frac{\Delta p_y}{p} \quad \text{---------- (3)}$$

$$\text{Equating (2) and (3) we get } \frac{\lambda}{\Delta y} = \frac{\Delta p_y}{p}$$

$$\text{or } \Delta y.\, \Delta p_y = \;= \lambda \times p = \frac{h}{p} X\, p \quad \text{(using de Broglie hypothesis)}$$

Hence $\Delta y.\ \Delta p_y = h$ ------ Heisenberg's uncertainty principle.

Q24. State Uncertainty Principle. Write its mathematical form for the following pairs of variables: (i) Position and momentum, (ii) Energy and time, (iii) Angular position and angular momentum. (4) S-01

Ans.

Heisenberg's Uncertainty principle: It is impossible to measure simultaneously both position and momentum of a particle with unlimited accuracy.

(i) $\Delta x.\ \Delta p_x \geq \hbar$.

(ii) $\Delta E.\Delta t \geq \hbar$

(iii) $\Delta\Theta.\Delta L \geq \hbar$

Application/Consequences of Heisenberg's uncertainty principle –i) It is significant only for microscopic particles.
ii) Electron cannot exist in nucleus

Q26. State Heisenberg Uncertainty Principle and Prove that electron cannot be present inside nucleus of an atom. (4) W-16, W-17, W-12

Q27. State Heisenberg's uncertainty principle and prove that electrons are not the constituents of nucleus. (4) W-08

Q28. Show that electron cannot be present inside nucleus on the basis of Uncertainty Principle. (3) S-18

Q29. Prove that electron cannot exist in nucleus. The radius of the nucleus =10^{-14} m. and maximum kinetic energy of electron in an atom is 4 MeV. (3) S-13

Q30. Using the following data show that electron cannot exist in nucleus. Given:The radius of the nucleus is 10^{-14}m and maximum kinetic energy of electron in an atom is 4 MeV. (3) S-14, S-16

Ans. Heisenberg's Uncertainty principle: It is impossible to measure simultaneously both position and momentum of a particle with unlimited accuracy.

To prove that electron cannot exist in nucleus: Let us assume that electron exists in nucleus.

Radius of the nucleus of an atom is 'r' $\cong 10^{-14}$m

Hence diameter of nucleus'd'= 2 x 10^{-14}m

If an electron is inside the nucleus then the maximum uncertainty in its position will be equal to this diameter

$$\therefore \Delta x = 2 \times 10^{-14} m$$

By Heisenberg's uncertainty principle,

$$\Delta x . \Delta p_x \geq \hbar$$

$$\therefore (\Delta p_x)_{minimum} = \frac{\hbar}{\Delta x} = \frac{6.63 X 10^{-34}}{2\pi \ X \ 2X10^{-14}} = 0.528 \times 10^{-20} kg.m/s$$

Since the minimum value of momentum of electron must be equal to $(\Delta p_x)_{minimum}$

$$(\Delta p_x)_{minimum} = p_{minimum} = 0.528 \times 10^{-20} kg.m/s$$

Since $E_{minimum} \approx p_{minimum}\, c$

$= 0.528 \times 10^{-20} \times 3 \times 10^{8}$

$= 1.584 \times 10^{-12} J = \frac{1.584X10^{-12}}{1.602X10^{-19}} = 9.88 \ X10^{6}$ eV

$= 9.88 MeV \approx 10 MeV$

But 10MeV >>>>4MeV(maximum kinetic energy of electrons orβ particles)

Therefore our assumption is wrong. Electron cannot exist in the nucleus.

Wave function and its probability interpretation.
Physical Significance of Ψ and mathematical conditions imposed on Ψ.

Q27.What is the physical significance of wave function Ψ. (3) W-14, W-15,W-16,S-05,W-05,S-06,S-07,W-12

Q28. Define Wave function. What is meant by normalized wave function? Explain in brief the Mathematical conditions imposed on Wave function. (1+1+3)W-11

Q29. State the Properties of wave function 'Ψ'. (2)W-13, S-16

Q30. What is the physical significance of wave function 'ψ' Explain in brief the mathematical conditions imposed on 'ψ'. (4)W-02

Ans. Wave function Ψ:

Wave variable which represents the de Broglie waves or the matter waves associated with a particle is called wave functionΨ .

Physical Significance of Ψ:

Just as light waves are represented by electric and magnetic field variations and sound waves by pressure variations, De Broglie matter waves can be represented by wave variable called wave function Ψ. **Mathematically Ψ describes the motion of particle in terms of position and time coordinates x, y, z and t.**

Physical significance of wave-function.

1] ψ as such has no direct physical significance as it is not a observable quantity. (Mathematically it describes the motion of an electron)

2] It is usually a complex quantity. (Consisting of real and Imagnary parts $x+iy$).

It may be expressed in the form of $\psi(x,y,z,t) = a+ib$. Where a & b are real functions of the variable (x,y,z,t) and $i=\sqrt{-1}$.

The complex conjugate of ψ which is denoted by ψ^* is obtained by changing to $-i$.

$\psi^*(x,y,z,t) = a-ib$

Although ψ itself is not a measurable quantity, all measureable quantities such as energy, momentum of a particle can be described from knowledge of ψ.

3] Mathematically ψ represents the motion of the particle like an electron.

4] In general ψ is a function of x,y,z,t i.e space & time co-ordinates. But it is not possible to locate position of the particle precisely at a position (x,y,z,t). There is only a probability of a finding a particle being at a specific point (x,y,z).

5] $|\psi|^2$ has physical significance i.e If $\psi(x,y,z)$ represents a particle, then square of the absolute value of the wave-funct. $|\psi|^2$ is proportional to the probability of a particle being in ~~[illegible]~~ unit volume of space, centred at the point (x, y, z), where ψ is to be evaluated at time t.

$|\psi|^2$ has a physical significance.

If $dv = dx\,dy\,dz$ is an infinitesimally small volume element surrounding the point (x, y, z) then the probability of find the particle in that volume is proportional to $|\psi|^2 dv = |\psi|^2 dx\,dy\,dz$

According to Heisenberg's uncertainty principle, there exists an uncertainty in finding the exact position of the particle at a certain point (x, y, z) in space. Therefore, Ψ becomes a complex quantity which is not observable and has no direct physical significance.

According to ***Max Born***, to make Ψ real, we can multiply it by its complex conjugate Ψ^*. Hence $\Psi.\Psi^*=|\Psi|^2$ is the probability density or probability per unit volume of finding a particle in a given region of space. Thus for a infinitesimally small volume element $dV = dx.dy.dz$, if 'P' is the probability of finding the particle in volume element dV, then

$$P \alpha |\Psi|^2 dV$$

Normalization condition

Q. Why normalization of ψ is required?

If at all particle exists. The probability of finding the particle somewhere in the universe must be unity. Since the probability of its being located in an elemental volume is proportional $\propto |\psi|^2 dx\,dy\,dz$,

It is convenient to choose the constant of proportionality such that the sum of probabilities over all values of x, y, z must be unity. Thus

$$\int_{x=-\infty}^{x=+\infty}\int_{y=-\infty}^{y=+\infty}\int_{z=-\infty}^{z=+\infty} \psi\,\psi^* \,dx\,dy\,dz = 1$$

or

$$\iiint_{-\infty}^{+\infty} |\psi|^2 dx\,dy\,dz = 1$$

This is known as normalization condition.

The wave-function which satisfies this condition is known as normalized wave-function.

{Since the particle is certainly to found somewhere in space, the integral of $|\Psi|^2 dV$ over the complete space must be equal to unity.

$$\int_{-\infty}^{+\infty} |\Psi|^2 dV = 1 \text{ -------- (1)}$$

Condition (1) is called ***normalization condition*** *and a wave function which satisfies this condition is called* ***normalized wave function****. When the wave function is normalized,*

$$P = |\Psi|^2 dV$$

To normalize a wave function, we multiply it by a constant factor such that condition (1) is satisfied.}

Mathematical conditions imposed on Ψ /Properties of wave function Ψ:

In order to make Ψ a well behaved wave function certain mathematical conditions are imposed on it. These conditions are:

1) Ψ should be a **normalized wave function**.

2) **Ψ(x, y, z) should be finite** even if values of x, y, z are infinite. If Ψ is not finite, then probability of finding the particle in space becomes infinite. This contradicts Heisenberg's uncertainty principle.

3) **Ψ(x, y, z) and its space derivatives should be continuous** across any boundary where the potential changes. Since Ψ(x, y, z) is related to a real quantity (i.e. particle) hence it cannot be discontinuous across any boundary.

4) **Ψ(x, y, z) should be single valued at any point.** If Ψ has more than one value at any point, then the probability of finding the particle will have more than one value at a given point. This is not possible as Ψ is related to a real quantity.

Schrodinger's Time dependent & time independent equations.

Q32. Write Schrodinger's time independent and time dependent wave equations in 3-D.
(2) W-17

Q33. Write Schrodinger's time independent and time dependent wave equations.
(2)S-08, W-10,W-05,S-05,S-00

Ans. Time dependant Schrodinger's wave equations

- $\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}$ + V Ψ = $i\hbar\frac{\partial\Psi}{\partial t}$ -------- one dimensional
- $\frac{\hbar^2}{2m}\nabla^2\Psi$ + V Ψ = $i\hbar\frac{\partial\Psi}{\partial t}$ -------- Three dimensional

where V is the applied potential, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ = differential operator and Ψ is the wave function associated with the particle of mass 'm'.

. Time independant Schrodinger's wave equations

- $\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}$ + V Ψ = EΨ -------- one dimensional
- $\frac{\hbar^2}{2m}\nabla^2\Psi$ + V Ψ = EΨ-------- Three dimensional

where 'E' is the total energy possessed by the particle. These equations describe behaviorof matter waves.

Application of Schrodinger's time independent Equation:

1. Solution of Schrodinger's equation for one dimensional infinite potential well.

Q34. Show that the energy of a micro particle confined in an infinite one-dimensional potential well of length 'L' is quantized. In the above situation, the particle cannot have zero energy. Explain, Why. **(5) S-13**

Q35. A free particle of mass "m" is kept in a rectangular box of length "L". Considering one dimensional motion, obtain an expression of discrete energy of particle. Show that energy of particle is quantized. **(5)W-13**

Q36. A particle is confined in a one dimensional potential well of infinite depth. Use the Schrodinger wave equation to obtain energy states of a particle inside the well. (3) W-14

Q37. Show that the wavefunction for a particle confined to move in an infinite one-dimensional potential well of length '*l*' is given by, En = n^2h^2 / $8ml^2$ where symbols have their usual meanings. Is the electron trapped in a potential well allowed to have zero energy? Why?
(5) S-15,W-11,S-08,S-07,S-03

Q38. Using Schrödinger's time independent equation, obtain an expression for energy states of electron trapped in an infinite potential well of width L. **(5) W-16, S-12,S-11**

Q39.Obtain an expression for quantized energy for an electron trapped in one dimensional potential well of infinite height of width L. **(4) S-18**

Q40. Show that the solution of Schrodinger Equation for a particle in an infinite potential well leads to the concept of Quantization of energy. **(4) W-06**

Ans.

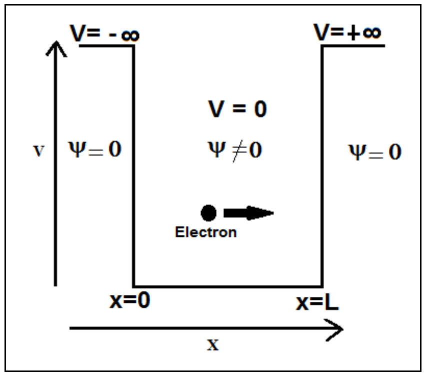


Figure 4: Electron in infinite one-dimensional potential well of width 'L'

Consider an electron confined to move within a one dimensional potential well of infinite height and width 'L' as shown in figure 4. The electron is trapped in the well and has potential energy so low that it cannot escape from the well. The electron moves in x-direction and reflects from the boundary walls of potential well.Thus potential energy (V) of the electron is treated as zero. Therefore, the boundary conditions are,

$$\Psi(x) = 0 \text{ for } x \leq 0 \text{------- (1)}$$

$$\text{and } \Psi(x) = 0 \text{ for } x \geq L \text{-------(2)}$$

To describe the motion of electron inside the potential well, we use one dimensional Schrodinger's time independent wave equation which is given by

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi = E\Psi$$

Since V=0 inside the potential well, hence the above equation reduces to

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} = E\Psi$$

$$or\ \frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + E\Psi = 0$$

$$\text{or } \frac{\partial^2\Psi}{\partial x^2} + \frac{2mE}{\hbar^2}\Psi = 0$$

Since $\hbar = \frac{h}{2\pi}$, above equation becomes $\frac{\partial^2\Psi}{\partial x^2} + 4\pi^2 x \frac{2mE}{h^2}\Psi = 0$

$$\text{or } \frac{\partial^2\Psi}{\partial x^2} + \frac{8\pi^2 mE}{h^2}\Psi = 0 \text{ ---------- (3)}$$

$$\text{Substituting } k^2 = \frac{8\pi^2 mE}{h^2} \text{--------- (4)}$$

Then eqn. (3) takes the form $\frac{\partial^2\Psi}{\partial x^2} + k^2\Psi = 0$ ---------- (5)

General solution of such eqn. (5) can be written as $\Psi(x) = A\sin kx + B\cos kx$ ------- (6)
Where, A and B are constants, their values can be obtained by applying the boundary conditions. As the particle is enclosed between the two rigid walls, the probability of finding the particle outside is zero. Hence

(i) Using boundary condition (1) i.e. $\Psi(x) = 0$ for x=0 in eqn.(6) we get

$$0 = A\sin k.0 + B\cos k.0$$

$$\text{or } B = 0\ .$$

By considering B=0, eqn. (6) takes the form $\Psi(x) = A\sin kx$------ (7)
Using the boundary condition (2)i.e. for x=L, $\Psi(L) = 0$ in above equation(7) we get,

$$0 = A\sin kL$$

$$\text{Since } A \neq 0,\ \sin kL = 0$$

or kL = nπ where n= 1,2,3------------

or $k = \frac{n\pi}{L}$

Substitute this value of k in eqn.(4) we get, $k^2 = \frac{8\pi^2 mE}{h^2} = \frac{n^2\pi^2}{L^2}$

$$\text{or } E_n = \frac{n^2h^2}{8mL^2} \text{ -------- (8)}$$

From this expression it is clear that the energy values of the electron inside an infinite deep potential well cannot have any value, but only discrete values of energy for n=1,2,3------ where 'n' is a quantum number.

For n=1, $E_1 = \frac{1^2h^2}{8mL^2} = \frac{h^2}{8mL^2}$,

For n=2, $E_2 = \frac{2^2h^2}{8mL^2} = \frac{4h^2}{8mL^2}$,

For n=3, $E_3 = \frac{3^2h^2}{8mL^2} = \frac{9h^2}{8mL^2}$

$E_1, E_2, E_3 \ldots E_n$ are the ***Eigen values*** **or allowed values of the energy of the particle.** Thus, the energy values for an electron confined in an infinite one-dimensional potential well are quantized.

Electron cannot take zero value of energy:

The energy of electron cannot have zero value. If energy takes zero value, its momentum also would be zero.**{E= (½)mv^2= (1/2m) m^{2}v^2= p^2/2m.}**. From the uncertainty principle its wavelength 'λ' (λ=h/p)will then become infinite. Thus, the electron will no longer be confined inside the potential well. Therefore, electron must possess a certain minimum amount of kinetic energy $=\frac{h^2}{8mL^2}$. It is also known as Zero level/Zero point energy.

Q41.Show that wave function for a particle confined in an infinite one-dimensional potential well oflength "L" is given by $\sqrt{\frac{2}{L}}$ sin(nπx / L).Hence discuss the energy levels and their discreteness. (4)S-14,W-01

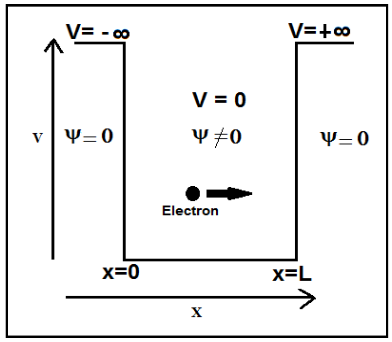
Q42 Using Schrödinger's time independent equation, obtain an expression for eigen function of particle in one dimensional potential well of infinite height. (5)W-15,W-12,W-08,W-00

Q43. Show that the wave function for a particle confined in a one dimensional potential well of length L is given by Ψn (x) = A sin(nπx / L). (5) S-16

Q44.Derive an expression for wavefunction of an electron confined to move in infinite potential well of width L. (5) S-17,S-10

Q45.Show that the wave function for a particle confined in a one dimensional potential well of length L and infinite depth is given by Ψn (x) = A sin(nπx / L). Hence using normalization condition on Ψshow that A is given by √ 2/L. (5)W-17,W-09,W-07

Ans.



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Consider an electron confined to move within a one dimensional potential well of infinite height and width 'L' as shown in figure 4. The electron is trapped in the well and has potential energy so low that it cannot escape from the well. The electron moves in x-direction and reflects from the boundary walls of potential well. Thus potential energy (V) of the electron is treated as zero. Therefore, the boundary conditions are,

$$\Psi(x) = 0 \quad \text{for} \quad x \leq 0 \text{------- (1)}$$

$$\text{and } \Psi(x) = 0 \quad \text{for} \quad x \geq L \text{ -------(2)}$$

To describe the motion of electron inside the potential well, we use dimensional Schrodinger's time independent equation which is given by

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\,\Psi = E\Psi$$

Since V=0 inside the potential well, hence the above equation reduces to

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} = E\Psi$$

$$or\ \frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + E\Psi = 0$$

$$\text{or } \frac{\partial^2 \Psi}{\partial x^2} + \frac{2mE}{\hbar^2}\Psi = 0$$

Since $\hbar = \frac{h}{2\pi}$, above equation becomes $\frac{\partial^2 \Psi}{\partial x^2} + 4\pi^2 x \frac{2mE}{h^2}\Psi = 0$

$$\text{or } \frac{\partial^2 \Psi}{\partial x^2} + \frac{8\pi^2 mE}{h^2}\Psi = 0 \text{ ---------- (3)}$$

$$\text{Substituting } k^2 = \frac{8\pi^2 mE}{h^2} \text{ --------- (4)}$$

Then eqn. (3) takes the form $\frac{\partial^2 \Psi}{\partial x^2} + k^2\,\Psi = 0$ ---------- (5)

General solution of such eqn. (5) can be written as $\Psi(x) = A \sin kx + B \cos kx$ ------- (6)

Where, A and B are constants, their values can be obtained by applying the boundary conditions. As the particle is enclosed between the two rigid walls, the probability of finding the particle outside is zero. Hence

(i) Using boundary condition (1) i.e. Ψ(x) =0 for x=0 in eqn.(6) we get

$$0 = A \sin k.0 + B \cos k.0$$

$$\text{or } B = 0 .$$

By considering B=0, eqn. (6) takes the form $\Psi(x) = A \sin kx$ ------ (7)

Using the boundary condition (2) i.e. for x=L, Ψ (L) = 0 in above equation (7) we get,

$$0 = A \sin kL$$

$$\text{Since } A \neq 0,\ \sin kL = 0$$

$$\text{or } kL = n\pi \quad \text{where } n = 1,2,3\text{------------}$$

$$\text{or } k = \frac{n\pi}{L}$$

Substitute this value of k in eqn.(4) we get, $k^2 = \frac{8\pi^2 mE}{h^2} = \frac{n^2\pi^2}{L^2}$

$$\text{or } E_n = \frac{n^2h^2}{8mL^2} \text{ -------- (8)}$$

E_1, E_2, E_3.....E_n are the ***Eigen values*** or allowed values of the energy of the particle. Thus the energy values for an electron confined in an infinite one-dimensional potential well are quantized.
Since $k = \frac{n\pi}{L}$, Eqn. (7) becomes $\Psi(x) = A \sin kx = A \sin \frac{n\pi}{L} x$----(9)

To find the value of 'A'we use normalization condition,

$$\int_{x=0}^{x=L} |\Psi|^2 dx = 1$$

$$\int_{x=0}^{x=L} |A \sin \frac{n\pi}{L} x|^2 dx = 1$$

$$\text{or} |A|^2 \int_{x=0}^{L} sin^2 (\frac{n\pi x}{L}) dx = 1$$

{because $\cos 2x = 1 - 2\sin^2 x$, hence $\sin^2 x = \frac{1}{2}(1 - cos2x)$}

$$\text{or } \frac{|A|^2}{2} \int_{x=0}^{L} (1 - cos2\frac{n\pi}{L} x)\, dx = 1$$

$$\text{or } \frac{|A|^2}{2} \left\{ \int_{x=0}^{L} 1dx - \int_{x=0}^{L} cos2\frac{n\pi}{L} xdx \right\} = 1$$

$$\text{or } \frac{|A|^2}{2} \ (x)_{x=0}^{L} = 1$$

$$\text{or } \frac{|A|^2}{2} (L - 0) = 1 \text{ or } |A|^2 = \frac{2}{L} \text{ or } A = \sqrt{\frac{2}{L}}$$

$$\text{Therefore, } \Psi_n(x) = A \sin kx = A \sin \frac{n\pi}{L} x = \sqrt{\frac{2}{L}} \sin \frac{n\pi}{L} x \ .$$

The above equation gives the wave function for a particle confined in an infinite one-dimensional potential well of length 'L'. $\Psi_n(x)$ are called ***eigenfunctions*** associated with electron.

Numericals:

Problems based on Heisenberg's Uncertainty Principle:
Formulae:
1. $\Delta x . \Delta p_x = \hbar$
2. $\Delta x . m\Delta v_x = \hbar$
3. $\Delta v = \frac{\hbar}{m\Delta x}$

1. A proton is confined to a nucleus of radius 10^{-14}m. Calculate the minimum uncertainty in velocity of the proton. (3) S-14

Ans. **Given:** L= 10^{-14} m, h= 6.63X10^{-34}Js, mass of proton 'm'= 1.67X10^{-27}kg, Radius of nucleus = 10^{-14}m , Δv=?

Solution: Since Radius of nucleus 'r' = 10^{-14}m,Δx = 2r = 2X10^{-14} m

$$\Delta v = \frac{\hbar}{m\Delta x} = \frac{h}{2\pi m\Delta x} = \frac{6.63X10^{-34}}{2X3.14X1.67X10^{-27}X2X10^{-14}} = 3.14X10^{6}m/s.$$

2. An electron has a speed of 600 m/s with an accuracy of 0.005%.Calculate the uncertainty with which we can locate the position of electron. (2) S-15

Ans. Given: v=600 m/s, Δv= 0.005% of v, h= 6.63X10^{-34}Js , m=9.1X10^{-31}kg , Δx = ?

Solution: Δv= 0.005% of 600m/s= 0.03

$$\Delta x = \frac{\hbar}{m\Delta v} = \frac{h}{2\pi m\Delta v} = \frac{6.63X10^{-34}}{2X3.14X9.1X10^{-31}X0.03} = 3.86X10^{-3}m=3.86mm$$

3. Compute the minimum uncertainty in the location of body having mass of 2 gm moving with a speed of 1.5m/s and the minimum uncertainty in the location of an electron moving with a speed of 0.5 X 10^{8} m/s. Given that the uncertainty in the momentum p for both is Δp=10^{-3}p. (3) W-15, W-09

Ans. For body of mass 2gm: m=2gm=2X10^{-3}kg, v =1.5m/s, h= 6.63X10^{-34}Js,Δx = ?

Solution: Since Δp=10^{-3}p , Δp=10^{-3}mv= 10^{-3}X 2X10^{-3}X1.5=3X10^{-6}kgm/s

$$\Delta x.\Delta p_x = \hbar$$

$$\Delta \mathbf{x} = \frac{\hbar}{\Delta p_x} = \frac{\boldsymbol{h}}{2\pi\Delta p_x} = \frac{6.63X10^{-34}}{2X3.14X3X10^{-6}} = 0.351X10^{-28}m = 3.59X10^{-29}m$$

For Electron: m=9.1X10^{-31}kg, v =0.5X10^{8}m/s, h= 6.63X10^{-34}Js,Δx = ?

Solution: Since Δp=10^{-3}p, Δp=10^{-3}mv= 10^{-3}X 9.1X10^{-31}X0.5X10^{8}=4.55X10^{-26}kgm/s

$$\Delta x.\Delta p_x = \hbar$$

$$\Delta \mathbf{x} = \frac{\hbar}{\Delta p_x} = \frac{\boldsymbol{h}}{2\pi\Delta p_x} = \frac{6.63X10^{-34}}{2X3.14X4.55X10^{-26}} = 2.3X10^{-9}m$$

4. An electron and a bullet (50gms) are travelling with same velocity of 300m/s. Assuming accuracy of 0.01% in velocity measurement, calculate the accuracy in location of their positions. (3) W-17,W-12

Ans. For bullet:

Given: v=300 m/s, Δv= 0.01% of v, h= 6.63X10^{-34}Js , mass 'm'= 50gms=50X10^{-3}kg , Δx = ?

Solution: Δv= 0.01% of 300m/s= 0.03

$$\Delta x = \frac{\hbar}{m\Delta v} = \frac{h}{2\pi m\Delta v} = \frac{6.63X10^{-34}}{2X3.14X50X10^{-3}X0.03} = 7X10^{-32}m$$

For Electron:

Δv= 0.01% of 300m/s= 0.03, m=9.1X10^{-31}kg , Δx = ?

$$\Delta x = \frac{\hbar}{m\Delta v} = \frac{h}{2\pi m\Delta v} = \frac{6.63X10^{-34}}{2X3.14X9.1X10^{-31}X0.03} = 3.86X10^{-3}m=3.86mm$$

5.Show that the uncertainty in measurement of electron momentum is equal to its momentum itself; if the uncertainty in measurement of location of electron is equal to de-Broglie wavelength. (3) W-11,S-02

Ans.

Given:Uncertainty in measurement of location of electron 'Δx'= de-Broglie wavelength 'λ'

To Show that Uncertainty in measurement of electron momentum 'Δp' = p

According to Heisenberg's Uncertainty Principle

$$\Delta x.\Delta p = h/2*3.14$$

$$\therefore \Delta p = \frac{h}{\Delta x} = \frac{h}{\lambda} \text{ --------- (Given: } \Delta x = \lambda)$$

$$\text{But de-Broglie wavelength } \lambda = \frac{h}{p}$$

$$\therefore p = \frac{h}{\lambda}$$

$$\text{Hence } \Delta p = \frac{h}{\Delta x} = \frac{h}{\lambda} = p$$

Problems based on quantization of Energy

1. $E_n = \frac{n^2h^2}{8mL^2}$ **2. For n=1,** $E_1 = \frac{1^2h^2}{8mL^2} = \frac{h^2}{8mL^2}$ **3. For n=2,** $E_2 = \frac{2^2h^2}{8mL^2} = \frac{4h^2}{8mL^2} = 4E_1$

4. For n=3, $E_3 = \frac{3^2h^2}{8mL^2} = \frac{9h^2}{8mL^2} = 9E_1$

5. $\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mE}}$ **6. 1eV= 1.602x10^{-19}Coulomb**

6. Calculate the lowest three permissible energies of an electron if it is bound by an infinite square potential well of width 2.5 x10^{-10} m. (3) W-13

Ans. Given: L= 2.5 x10^{-10} m, h= 6.63X10^{-34}Js, mass of electron 'm'= 9.1X10^{-31}kg

Solution: $E_n = \frac{n^2h^2}{8mL^2}$

For n=1, $E_1 = \frac{h^2}{8mL^2} = \frac{(6.63X10^{-34})^2}{8X9.1X10^{-31}X(2.5X10^{-10})^2} = 9.66X10^{-19}$ J= $\frac{9.66X10^{-19}}{1.602X10^{-19}}$ =6.029 eV

$E_2 = 4E_1 = 4X9.66\ X10^{-19}$J = 38.64X 10^{-19} J = 24.11 eV

$E_3 = 9E_1 = 9X9.66\ X10^{-19}$J =86.94X$10^{-19}$ J =54.26 eV

7. Find the lowest three energy of an electron confined to move in a one dim. box of length 5 Á. (3)W-14,W-17,S-16,W-07

Ans. Given: L=5 ´ = 5 x10^{-10} m, h= 6.63X10^{-34}Js, mass of electron 'm'= 9.1X10^{-31}kg, E_1=?, E_2=?,E_3=?

Solution: $E_n = \frac{n^2h^2}{8mL^2}$

For n=1, $E_1 = \frac{h^2}{8mL^2} = \frac{(6.63X10^{-34})^2}{8X9.1X10^{-31}X(5X10^{-10})^2} = 2.415X10^{-19}$J= $\frac{2.415X10^{-19}}{1.602X10^{-19}}$ = 1.5eV

$E_2 = 4E_1 = 4X\ 2.415\ X10^{-19}$J = 9.66X 10^{-19} J = 6 eV

$E_3 = 9E_1 = 9X\ 2.415\ X10^{-19}$J =21.73X$10^{-19}$ J = 13.5eV

8. A baseball of mass 1 mg is confined to move between two rigid walls separated by 1 cm. Calculate its lowest energy and minimum speed. (3)W-10

Ans.

Given: mass of baseball 'm'= 1mg=10^{-3}gm= 10^{-}3x10^{-3}kg = 10^{-6}kg, L=1cm =10^{-2} m, $E_{min.}=E_1$=? $v_{min.}$= ?

Solution: $E_{min.} = E_1 = \frac{h^2}{8mL^2} = \frac{(6.63X10^{-34})^2}{8X10^{-6}X(10^{-2})^2}$ = 5.49X 10^{-58}J

$$E_{min.} = \frac{1}{2}mv^2$$

$$v^2 = \frac{2E_{min.}}{m}$$

$$v = \sqrt{\frac{2E_{min.}}{m}} = \sqrt{\frac{2X5.49X10^{-58}}{10^{-6}}} = 3.31X10^{-26}\text{m/s}$$

9.Find the lowest energy and momentum of an electron in one dimensional potential well of width 1Á. Express the result in eV. (3)S-10

Ans. Given: L=1 ´ = 10^{-10} m, h= $6.63X10^{-34}$Js, mass of electron 'm'= $9.1X10^{-31}$kg, E_1=?,p=?

Solution:$E_n = \frac{n^2h^2}{8mL^2}$

For minimum energy, n=1,∴$E_1 = \frac{h^2}{8mL^2} = \frac{(6.63X10^{-34})^2}{8X9.1X10^{-31}X(10^{-10})^2}$ =$60.1X10^{-19}$J

$= \frac{60.1X10^{-19}}{1.602X10^{-19}}$ =37.5eV

Since $E = \frac{p^2}{2m}$ Therefore $p = \sqrt{2mE_1} = \sqrt{(2X9.1X10^{-31}X60.1X10^{-19})}$ = $3.3X10^{-24}$kgm/s

10. An electron is confined to move between two rigid walls separated by 1nm. Find the de Broglie wavelengths representing the first three allowed energy levels of the electron and the corresponding energy. **(3) S-09, S-01**

Ans.

Given: L=1 nm = 10^{-9} m, h= $6.63X10^{-34}$Js, mass of electron 'm'= $9.1X10^{-31}$kg,

E_1=?,E_2=?, E_3=?, λ_1=?, λ_2=?, $\boldsymbol{\lambda}_3$=?

Solution:$E_n = \frac{n^2h^2}{8mL^2}$

For minimum energy, n=1,∴$E_1 = \frac{h^2}{8mL^2} = \frac{(6.63X10^{-34})^2}{8X9.1X10^{-31}X(10^{-9})^2}$ =$6.03X10^{-20}$J

$\lambda_1 = \frac{h}{p_1} = \frac{h}{\sqrt{2mE_1}} = \frac{6.63X10^{-34}}{\sqrt{2X9.1X10^{-31}X6.03X10^{-20}}}$ = $2X10^{-9}$m

E_2 =$4E_1$= $4X6.03X10^{-20}$J = $24.12X10^{-20}$J

$\lambda_2 = \frac{h}{p_2} = \frac{h}{\sqrt{2mE_2}} = \frac{6.63X10^{-34}}{\sqrt{2X9.1X10^{-31}X24.12X10^{-20}}}$= $1X10^{-9}$m

E_3 =$9E_1$= $9X6.03X10^{-20}$J = $54.27X10^{-20}$J

$\lambda_2 = \frac{h}{p_2} = \frac{h}{\sqrt{2mE_2}} = \frac{6.63X10^{-34}}{\sqrt{2X9.1X10^{-31}X54.27X10^{-20}}}$= $0.66X10^{-9}$m

1. Calculate the Zero-point energy for a particle in an infinite potential well for an electron confined to a 1 nm atom.
   a) 3.9 X 10^{-29} J
   b) 4.9 X 10^{-29} J
   c) 5.9 X 10^{-29} J
   d) 6.9 X 10^{-29} J

Ans. (c)

Explanation: Here, m = 9.1 X 10^{-31} kg, L = 10^{-9}m.

Therefore, $E = \frac{h^2}{8mL^2}$

$$E = \frac{(6.626 \times 10^{-34})^2}{8 \times 9.11 \times 10^{-31} \times (10^{-9})^2}$$

= 5.9 X 10^{-29} J

2. Calculate the minimum uncertainty in the velocity of an electron confined to a box of 10^{-8} m length.
   a) 601.45 m/s
   b) 16015 m/s
   c) 11601 m/s

d) 1161.45 m/s

Ans. (c)

Explanation: Here, $\Delta x = 10^{-8}$ m, $m = 9.11\text{x}10^{-31}$ kg,

$$\text{Therefore } \Delta v_x = \frac{\text{h}}{2\pi\text{m}\Delta x}$$

$$= \frac{6.63 \times 10^{-34}}{2 \times X3.14 \times 9.1X10^{-31} \times 10^{-8}}$$

$$= 11601.45\ m/s$$

3. Consider that an electron is confined to a box of length 10^{-10} m. Calculate the lowest energy of the system.
   a) 37.6 eV
   b) 37.6 J
   c) 73.5 eV
   d) 73.5 eV

Ans. (a)

Explanation: Here, $\text{L} = 10^{-10}\text{m}$, Mass of electron 'm' $= 9.1 \times 10^{-31}$ Kg

$$\text{Therefore } \text{E}_\text{n} = \frac{\text{n}^2\text{h}^2}{8\text{mL}^2}$$

For lowest energy, n = 1,

$$\therefore\ \text{E}_1 = \frac{\text{h}^2}{8\text{mL}^2} = \frac{(6.63 \times 10^{-34})^2}{8 \times 9.1 \times 10^{-31} \times (10^{-10})^2}$$

$$= 6.03 \times 10^{-18}\ \text{J}$$

$$= \frac{6.03 \times 10^{-18}}{1.602 \times 10^{-19}} = 37.6\ \text{eV}$$

Problems for Practice:

1. An electron is confined to move in a one dimensional potential well of length 10^{-10}m, find the quantized energy value for the lowest energy state. (2)S-13

Ans. $E_1 = 6.03\text{x}10^{-18}\text{J} = 37.5\text{eV}$

2. An electron is trapped in one dimensional potential well of finite depth and width 1mm. Calculate first three lowest energy values. (3)S-14

Ans. $E_1 = 6.03\text{x}10^{-32}\text{J} = 3.76\ \text{x}10^{-13}\text{eV}$, $E_2 = 24.12\text{x}10^{-32}\text{J} = 1.505\text{x}10^{-12}$ eV,
$E_3 = 54.27\text{X}10^{-32}\text{J} = 3.387\text{x}10^{-12}\text{eV}$

3.Find two lowest energy states of an electron trapped in an infinite potential well of width 2 Á. Express results in eV. (3) W-16,S-17

Ans. $E_1 = 1.509\text{x}10^{-18}\text{J} = 9.42\text{eV}$

4. An electron is confined to move in one dimensional potential well of length 7 Á. Find the quantized energy values for the three lowest energy states. (3) S-18

Ans. E_1= 1.232x10^{-19}J =0.769eV, E_2= 4.928x10^{-19}J = 3.076eV,

E_3= 1.108x10^{-18}J = 6.921 eV

5. Calculate the minimum uncertainty in the velocity of an electron confined to a box of 10^{-8} m length. (2) W-13

Ans. Δv= 0.1159x10^5m/s = 11593.406 m/s

6. Calculate minimum uncertainty in the velocity of an electron confined to a box of 10^{-10} m length. (2) W-16

Ans. Δv= 0.1159x10^7m/s = 1,159,340.659 m/s

7. A proton is confined to a nucleus of radius 10^{-14}m. Calculate the minimum uncertainty in velocity and momentum of the proton. (3) S-11

Ans. Δv=3.14x10^6m/s, Δp= 5.624x10^{-21} kgm/s

8. Calculate the minimum uncertainty in the location of a mass of 5 gm moving with a speed of 2m/s and the minimum uncertainty in the location of an electron moving with a speed of 6 x10^7m/s. Given that the uncertainty in the momentum p for both is Δ p = 10^{-3}p. (3)S-15

Ans. For mass of5gm, Δx = 1.055x10^{-29}m

For Electron, Δx = 1.932x10^{-9}m

9. Calculate uncertainty in location of an electron and a ball of mass 1kg if their velocities are 10^5 m/s and 10m/s respectively. (3) S-17

Ans. For ball, Δx = 1.055x10^{-35}m

For Electron, Δx = 1.159x10^{-9}m

10. An electron has a speed of 400m/s with an accuracy of 0.001%.Calculate the uncertainty with which we can locate the position of electron. (3) S-18

Ans. Δx =0.0289m